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A Precise Upper Limit for the Correctness of the Navier–Stokes Theory with Respect to the Kinetic Theory

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The condition of positive normal pressures must hold for all solutions in the kinetic theory, but is violated by the Navier–Stokes equations for sufficiently high distortions. A dimensionless measure of this discrepancy is furnished by the tension number. In order for all pressures to be positive, it is necessary and sufficient that the tension number be less than 1. If this condition is violated, the normal-stress effects of the kinetic and Navier–Stokes theories are of opposite sign.

KEY WORDS: kinetic theory of gases; Navier–Stokes theory of fluids; tension number; truncation number; Knudsen number; normal-stress effects.

It is well known that according to the kinetic theory of monatomic, moderately rarefied gases, the Navier–Stokes constitutive equation is only an approximation. While various arguments have been put forward to this effect, they all rest on pictorial remarks or on purely formal processes of approximation to solutions of the Maxwell–Boltzmann equation which belong to a hypothecated special class, the existence of which remains still a matter of conjecture. Even more than this, the arguments labor under confusion of the sufficient with the necessary. Typically, the author presents some calculation intended to convince the reader that certain assumptions in the kinetic theory reduce it to the Navier–Stokes theory, whereupon he proclaims that these assumptions *must*[2] hold *in order* for the Navier–Stokes theory to follow!

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[2] An exception is the work of Maxwell.(1) First, Maxwell makes no such converse claim. Second, from his Eqs. (121) and (75), a necessary and sufficient condition could be formulated, but it would

In fact, a simple *necessary* condition for the Navier–Stokes theory can be derived by rigorous, though trivial, mathematics, directly from the definitions and without use of the Maxwell–Boltzmann integrodifferential equation, let alone of any special process of integration of it. The key to the argument is that while the older studies always focused upon the shear stress, modern continuum mechanics has taught us to expect that departures from classical predictions manifest themselves first in normal-stress effects (see, for example, Truesdell and Noll⁽²⁾). The crux of the argument is that while the definition of the pressure tensor in the kinetic theory forces the normal pressure on *every* element of area in *every* flow to be positive[3] unless the mass density vanishes, the Navier–Stokes theory makes a tension result on elements which are suffering sufficiently violent distortion. When these two normal-stress effects are contrary, the underlying theories cannot square with each other.

Indeed, we know from the definitions⁽⁴⁾ that the Navier–Stokes theory can never follow from the kinetic theory except subject to the Stokes viscosity relation. Thus, for comparison, we need consider only the case when the pressure tensor $\mathbf{P}$ is given in terms of the stretching tensor $\mathbf{D}$ as follows[4]:

$$\begin{aligned}\mathbf{P} &= (p + \tfrac{2}{3}\mu \operatorname{Tr} \mathbf{D})\mathbf{1} - 2\mu\mathbf{D} \\ &= p\mathbf{1} - 2\mu\mathbf{D}_0\end{aligned} \tag{1}$$

where $\mathbf{D}_0$ is the deviator of $\mathbf{D}$, often called the distortion tensor, and where p and μ, the pressure and the shear viscosity, are positive functions of density and temperature.[5] Hence,

$$\mathbf{n} \cdot \mathbf{P}\mathbf{n} = p - 2\mu\mathbf{n} \cdot \mathbf{D}_0\mathbf{n} \tag{2}$$

Therefore, the normal pressure on the element normal to $\mathbf{n}$ is positive if and only if

$$2\mathbf{n} \cdot \mathbf{D}_0\mathbf{n} < p/\mu \tag{3}$$

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be very complicated and would not have any obvious interpretation either in terms of the mean free path, etc., or in terms of measurable quantities. [These two equations are exact, but Eq. (121) is valid only for Maxwellian molecules.] Likewise, in the voluminous papers of Boltzmann I have found no claim that any particular condition of the gas is a necessary one for the theory of linear viscosity to hold.

[3] While I cannot find this conclusion in Chapman and Cowling,⁽³⁾ it follows immediately from Eq. (2) of Section 2.31. We must distinguish here between the true kinetic theory and the Chapman–Enskog iterates, the status of which has never been made precise except in very special cases. The condition of positive normal pressures holds for *all exact solutions*. If the Chapman–Enskog process converges to a particular exact solution, then this condition holds *in the limit*, but it certainly does not hold automatically at each finite stage. For example, while it does hold at the zeroth stage, which corresponds to Eulerian hydrodynamics, it does *not* hold automatically at the first stage, which corresponds to the Navier–Stokes theory. Presumably, the results at that stage are of value only insofar as they approximate properties of certain exact solutions. Hence arises the problem solved in the text, to which the preceding sentences of this footnote could serve as an alternative introduction.

[4] This tensor and the other kinematical apparatus used here are defined and described by Truesdell and Toupin,⁽⁵⁾ for example. See Sections 82 and 83.

[5] In fact, $p = kn\theta$, where n is the number density and θ is the temperature, but we do not need this explicit formula until we come to the special case below [Eq. (10)].

So as to ensure that this inequality holds for all $\mathbf{n}$, we render the left-hand side a maximum by choosing $\mathbf{n}$ as a proper vector of $\mathbf{D}_0$ corresponding to the largest proper number. The proper numbers of $\mathbf{D}$ are the principal stretchings d_i, and the corresponding proper numbers of $\mathbf{D}_0$ are $d_i - (1/3)\operatorname{Tr}\mathbf{D}$. If we order the d_i so that $d_1 \geqslant d_2 \geqslant d_3$, then the greatest proper number of $\mathbf{D}_0$ is one-third of the sum of the two greatest orthogonal shearings, namely, $\frac{1}{3}[(d_1 - d_3) + (d_1 - d_2)]$, both summands being nonnegative. For an isochoric motion, and only for such, this greatest distortion reduces to d_1, the greatest stretching. If we define the dimensionless *tension number* $\mathfrak{Te}$ by

$$\mathfrak{Te} \equiv (2/3)(2d_1 - d_2 - d_3)(\mu/p) \tag{4}$$

then a necessary and sufficient condition that no element of area be suffering positive tension is $\mathfrak{Te} < 1$. (Of course, it is impossible that *all* elements suffer tension, since the mean pressure, from its definition, is positive. Thus, at least one principal stress is always a pressure, no matter what be the flow.)

As a corollary to the foregoing remarks, we see that *the Navier–Stokes constitutive equation cannot be consistent with the Maxwellian kinetic theory unless*

$$\mathfrak{Te} < 1 \tag{5}$$

The tension number $\mathfrak{Te}$ is defined in terms of gross quantities alone, and thus may be used in continuum theories or kinetic theories at pleasure. Similar, but not identical, dimensionless parameters are used to estimate the error in cutting off constitutive equations after the linear terms. One such parameter is the *truncation number* $\mathfrak{Tr}$,[6] defined as follows:

$$\mathfrak{Tr} \equiv [2(d_1^2 + d_2^2 + d_3^2)]^{1/2}\,(\mu/p) \tag{6}$$

It can be shown that

$$\mathfrak{Te} \leqslant (2/\sqrt{3})\,\mathfrak{Tr} \tag{7}$$

where equality holds if and only if $d_2 = d_3 = -(1/2)\,d_1$. Thus, a small value of $\mathfrak{Tr}$ implies a small value of $\mathfrak{Te}$, but the converse is not generally true. Indeed, in a uniform dilatation, $\mathfrak{Te} = 0$ always, but for given μ/p, the truncation number $\mathfrak{Tr}$ is proportional to the rate of expansion or contraction and thus may be arbitrarily large.

In a plane isochoric flow, $d_2 = 0$ and $d_3 = -d_1 \leqslant 0$. In this case, the distinction between the truncation number and the tension number disappears: $\mathfrak{Te} = \mathfrak{Tr}$. To visualize the results, consider a simple shearing: $u = \kappa y$, $v = 0$, $w = 0$, where κ is the amount of shearing, assumed positive, so $d_1 = \frac{1}{2}\kappa$. Then

$$\mathfrak{Te} = \mathfrak{Tr} = \mu\kappa/p \equiv \mathfrak{T} \tag{8}$$

say.

For a gas of Maxwellian molecules, a relaxation time τ for the shear pressures in a gas grossly at rest exists and is given by[7]

$$\tau = \mu/p \tag{9}$$

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[6] A number of this kind seems first to have been introduced by Truesdell.(8) See Eq. (13). Cf. also Section 120 of Truesdell and Noll.(2)

[7] See Eq. (130) of Maxwell.(1)

Thus, for a simple shearing of such a gas,

$$\mathfrak{T} = \kappa\tau \tag{10}$$

which can be interpreted as being the amount of shearing, measured in units of the relaxation time. Thus, for this special case, the precise limit stated by Ineq. (5) asserts that the Navier–Stokes constitutive relation must fail before the shearing becomes as great as the relaxation rate. For the particular gas in question, both μ and p are proportional to the temperature, and, in fact,

$$\mathfrak{T} = \frac{1}{3}\left(\frac{2}{mG}\right)^{1/2}\left(\frac{\kappa}{nA_2}\right) \tag{11}$$

where m is the mass of one molecule, G is the intermolecular force constant, n is the number density, and $A_2 = 1.37\ldots.$ For a given Maxwellian gas, then, the condition (5) cannot hold at any density if the shearing is too great, nor can it hold for any shearing if the gas is too thin, but this interpretation of condition (5) is not general, because the ratio of μ/p in Eqs. (4) and (6) for non-Maxwellian gases will be a function of temperature as well as density.

Since there is no such thing as a Maxwellian gas in nature, numerical estimates are difficult. For a Maxwellian gas, $\tau = \tau_0(\rho_0/\rho)$, where τ_0 and ρ_0 are a particular pair of values of τ and ρ. If for τ_0 we substitute the value of μ/p for air at standard conditions, then the value of κ which brings $\mathfrak{T}\mathfrak{e}$ to its critical value, 1, is given roughly by

$$\kappa_{\text{limit}} = \tfrac{1}{2}(\rho/\rho_0) \times 10^{10}\ \text{sec}^{-1} \tag{12}$$

Thus, for example, in a vacuum of 10^{-6} atm, $\kappa_{\text{limit}} = 5000\ \text{sec}^{-1}$. Heating the gas to 1000° C cuts down the value of κ_{limit} by a factor of about 3.

As its name "truncation number" suggests, $\mathfrak{T}\mathfrak{r}$ was introduced originally as a parameter in heuristic arguments for cutting off nonlinear terms in the constitutive equations in fluids. So far as I know, the only results of this kind rigorously derived up to now in the kinetic theory concern a Maxwellian gas which undergoes homothermal simple shearing, in which, as we have remarked, $\mathfrak{T}\mathfrak{r} = \mathfrak{T}\mathfrak{e} = \mathfrak{T}$. These results are[8]:

1. While the differences of the principal normal stresses are zero according to the Navier–Stokes theory, in the kinetic theory the ratio of one of them to the mean pressure is $2\mathfrak{T}^2 + O(\mathfrak{T}^4)$ for small $\mathfrak{T}$, while the ratio of the shear pressure to the mean pressure is $\mathfrak{T} - \frac{4}{3}\mathfrak{T}^3 + O(\mathfrak{T}^5)$, the first term being the result of the Navier–Stokes theory. (In fact, deviations from the Navier–Stokes theory are noticeable

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[8] See the paper of Truesdell,(7) Sections 34, 36, and 51. The first conclusion agrees with Burnett's formulas, which are derived by the Chapman–Enskog process. Normal-stress effects have been known in the kinetic theory of gases since 1879, but they appeared in the literature of "rheology" only much later. To simplify the statements in the text above, I have neglected entirely the oscillatory parts of the pressure system, since they are damped out as time goes on. For details, see Truesdell,(7) especially Eq. (33.2).

on a graph of ordinary size as soon as $\mathfrak{T} > 0.1$, and when $\mathfrak{T} = 1$, the Navier–Stokes shear pressure is almost twice as big as the true value, while the difference of normal pressures is far greater than the shear pressure, so the Navier–Stokes theory fails to give even a rough picture.)

2. The Chapman–Enskog process converges if $\mathfrak{T} < \sqrt{2}/3$ and diverges if $\mathfrak{T} > \sqrt{2}/3$.
3. The heat flux is unstable if $\mathfrak{T} > 3/\sqrt{2}$.

Returning to the general condition (5), we repeat that it is satisfied by any expansion equal in all directions, no matter what be the gas. This is not surprising, since Maxwell[9] showed that in such a flow, the kinetic gas behaves like an ideal perfect gas in adiabatic expansion, and the same thing is true for a Navier–Stokes gas that satisfies the Stokes relation. Therefore, not only is the necessary condition (5) satisfied, but also the two theories then do agree.

Finally, we remark that no matter what be the molecular model, condition (5) cannot possibly hold as a *general* consequence of the kinetic theory for all flows. Indeed, the theory places no restriction on the molecular density function F at some given instant other than that it be summable. At some arbitrary initial time, then, the zeroth, first, and second moments of F may be any fields whatever. Since, then, the theory itself does not force these moments to satisfy any *a priori* inequality, in particular it does not force them to satisfy condition (5). Some solutions may satisfy it; indeed, we know that some do. Some solutions may fail to satisfy it initially, but come to satisfy it as time goes on. In the particular case of homothermal simple shearing of a Maxwellian gas, all four quantities d_1, d_2, d_3, and μ/p are constants, so the flow either satisfies condition (5) at the outset or never satisfies it.

Every statement about the Navier–Stokes theory given here refers to the constitutive equation as it stands, not to the result of substituting it in the field equations and integrating so as to solve an initial-value problem. This is a far more delicate matter.[10]

APPENDIX

In the text above I have used no terms and concepts except those of Maxwell's formal kinetic theory and of general continuum mechanics. These terms and concepts are clearly defined. In works on the dynamics of rarefied gases, selected results from the former of these theories are often combined rather mysteriously with verbal pictures drawn in the elementary kinetic theory and with special and often approximate formulas from the Navier–Stokes and Euler theories of fluids. As a result, considerable literary latitude is left to the student, so it is dangerous to try to describe in the same terms anything really specific. For example, the Knudsen number $\mathfrak{K}$ is defined as the ratio of the mean free path L to a "characteristic length" l of the problem. The

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[9] Equation (108) of Maxwell,(1) in the case when $\beta = 1$, this being the only case in which the theory given there is correct, as was remarked later by Poincaré.(8)

[10] See Section 4.2 of Truesdell's paper.(7)

mean free path depends on the molecular distribution function F, which is always unknown for nontrivial problems. What length is "characteristic" is also equivocal. While some authors determine such a length in terms of the speed of sound, that speed is not only a scalar field rather than a constant, but also is a presently unknown functional of the unknown molecular distribution function, so that use of the Laplacian acoustic speed for it, while common, is merely formal, without any known physical relevance to the kinetic problem.

After these precautionary explanations, we can relate the truncation number $\mathfrak{Tr}$ to the Knudsen number $\mathfrak{K}$ used in much of the literature. Namely, if we take for L the mean free path defined by the kinematic viscosity μ/ρ through the formula $\mu = \frac{1}{2}\rho\bar{c}L$, if we take the mean molecular speed $\bar{c}$ as being that for a gas in equilibrium at some pressure and density typical of the flow, and if we take for l the ratio of the Laplacian speed of sound in an ideal gas at rest at the typical temperature to a typical value of $[2(d_1^2 + d_2^2 + d_3^2)]^{1/2}$, then we shall find that $\mathfrak{Tr} \approx \mathfrak{K}$, pretty nearly. In general, of course, $\mathfrak{Tr}$ is not a constant, but the value of a scalar field. (This identification shows that a large value of $\mathfrak{K}$ does *not* necessarily invalidate the Navier–Stokes theory, since in an adiabatic expansion $\mathfrak{K}$ may be as large as desired, yet the Maxwell, Navier–Stokes, and Euler theories agree precisely.)

The results given here must not be taken as bearing upon the question of whether a rarefied gas may be regarded as a continuum, since Maxwell himself showed(1) that *all* solutions in his formal, mathematical kinetic theory represent the gas, no matter how rarefied it may be, as a continuum. (Much of the literature on the dynamics of rarefied gases only compounds confusion by using the term "noncontinuum flow" to refer to anything asserted to be beyond the range of application of the Navier–Stokes theory.)

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